# Partial Metric Spaces

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*1: Adjustment of the metric axioms*

This talk is based on the reference [B&].

We are used to the following definition:

**1 Definition.** A *metric space* is a pair $(X, d)$, where $d : X \times X \to \mathbb{R}$ and:

M0: $0 \leq d(x, y)$ (nonnegativity),

M1: if $x = y$ then $d(x, y) = 0$ ($= \Rightarrow$ indistancy),

M2: if $d(x, y) = 0$ then $x = y$ (indistancy $\Rightarrow =$),

M3: $d(x, y) = d(y, x)$ (symmetry), and

M4: $d(x, z) \leq d(x, y) + d(y, z)$ (triangularity).

A *pseudometric space* is such a pair $(X, d)$ satisfying M0, M1, M3 and M4.

For a metric space, $x = y$ if and only if $d(x, y) = 0$. Later we retain M2 but drop M1, leading to the study of self-distances $d(x, x)$ which may not be zero. This is motivated by the experience of computer science, as discussed below.

We begin with an example of a metric space, and why nonzero self-distance is worth considering. Let $X = S^\omega = \{x : \omega \to S\}$, the set of all infinite sequences in a set $S$, and let $d_S : X \times X \to \mathbb{R}$ be defined by: $d_S(x, y) = \inf\{2^{-k}{:}x_i = y_i \text{ for each } i < k\}$. It can be shown that $(S^\omega, d_S)$ is a metric space.

But computer scientists must compute the infinite sequence $x$, that is, write a program to print $x_0$, then $x_1$, then $x_2$, and so on. Since $x$ is infinite, its values cannot be printed in finite time, so computer scientists care about its *parts*, the finite sequences $\langle\ \rangle$, $\langle x_0\rangle$, $\langle x_0, x_1\rangle$, $\langle x_0, x_1, x_2\rangle$, .... For each $k$, the finite sequence $\langle x_0, \ldots, x_k\rangle$ is that part of the infinite sequence produced so far. Each finite sequence is thus thought as a *partially computed* version of the infinite sequence $x$, which is *totally computed.*

Suppose the above definition of $d_S$ is extended to $S^*$, the set of all finite and infinite sequences over $S$. Then M0, M2, M3 and M4 still hold, But if $x$ is finite then $d_S(x, x) = 2^{-k} > 0$ for some $k < \infty$, since $x_j = x_j$ can hold only if $x_j$ is defined. Thus axiom M1 does not hold for finite sequences.

Thus the truth of the statement $x = x$ is unchallenged in mathematics, while in computer science its truth can only be asserted to the extent to which $x$ is computed. This article will show that rather than collapsing, the theory of metric spaces is actually expanded by dropping M1.

An example of the same sort arises from the fact that the location of a point can only be measured to a tolerance, thus rather than thinking of $x \in Y$, $Y$ a metric space, it is appropriate to think of the ball $N_r(x) = \{y : d(x, y) \leq r\}$, where $r$ tells how accurately we could measure the location of $x$.

*2: Partial Metric Spaces.*

Nonzero self-distance is thus motivated by experience from computer science. Here is our generalization of the metric space axioms M0-M4 to introduce nonzero self-distance so that familiar metric and topological properties are retained.

2 Definition. A *partial metric space* is a pair $(X, p)$, where $p : X \times X \to \mathbb{R}$ is such that

P0: $0 \leq p(x, x) \leq p(x, y)$ (nonnegativity and small self-distances),

P2: if $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (indistancy implies equality),

P3: $p(x, y) = p(y, x)$ (symmetry), and

P4: $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$ (triangularity) (see [Ma94]).

These axioms also yield an associated metric space: define $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$.

Then the axioms P0, P2, P3, and P4 for $p$, imply M0 – M4 for $d_p$. Note how $p(y,y)$ is included in P4 to insure that M4 will hold for $d_p$. Thus $(X, d_p)$ is a metric space.

Each partial metric space thus gives rise to a metric space with the additional notion of nonzero self-distance introduced. Also, a partial metric space is a generalization of a metric space; indeed, if an axiom P1: $p(x,x) = 0$ is imposed, then the above axioms reduce to their metric counterparts. Thus, a metric space can be defined to be a partial metric space in which each self-distance is zero.

It is often convenient to use P4 in the equivalent form $p(x,z)+p(y,y) \le p(x,y)+p(y,z)$.

Why should axiom P2 deserve the title *indistancy implies equality*? If $p(x,y) = 0$, then by P0 and P3 $p(x,x) = p(x,y) = p(y,y)$, so $x = y$ by P2.

We wish to find as many ways as possible in which partial metric spaces may be said to extend metric spaces. That is, to apply as much as possible the existing theory of metric spaces to partial metric spaces, and to see how the notion of nonzero self-distance can influence our understanding of metric spaces.

Here are three partial metric spaces: The set of sequences studied in the last section, $(S^*, d_S)$, is a partial metric space, where the finite sequences are precisely those having nonzero self-distance.

For a second example, a very familiar function is a partial metric: Let $\max(a,b)$ be the maximum of any two nonnegative real numbers $a$ and $b$; then max is a partial metric on $\mathbb{R}^+ = [0,\infty)$. (For P4, if $\max(a,b,c) = b$ then $\max(a,c)+b \le b+b = \max(a,b)+\max(b,c)$, if $\max(a,b,c) = a$, then $\max(a,c)+b = a+b \le \max(a,b)+\max(b,c)$; the case $\max(a,b,c) = c$ is similar to the second.)

For a third example, for any metric space $(X,d)$, let $F(X)$ be the set $X \times \mathbb{R}^+$ of *formal balls* in $X$: here $(x,r)$ is thought of and denoted by $N_r(x)$, and $p(N_r(x), N_s(y)) = d(x,y) + \max(r,s)$. Then the self-distance of $N_r(x)$ is $r$, and $p$ is a partial metric on $F(X)$ (for triagularity, $p(N_r(x), N_t(z)) = d(x,z) + \max(r,t) \le (d(x,y) + d(y,z)) + (\max(r,s) + \max(s,t) - s) = p(N_r(x), N_s(y)) + p(N_s(y), N_t(z)) - p(N_s(y), N_s(y)))$.

And so partial metric spaces demonstrate that although zero self-distance has always been taken for granted in the theory of metric spaces, it is not necessary in order to establish a mathematics of distance. What they do is introduce a symmetric metric-style treatment of the nonsymmetric relation *is part of,* which, as explained below, is fundamental in computer science.

For each partial metric space $(X,p)$ let $\sqsubseteq_p$ be the binary relation over $X$ such that $x \sqsubseteq_p y$ (to be read, $x$ *is part of* $y$) if $p(x,x) = p(x,y)$. Then $\sqsubseteq_p$ is a is a partial order.

Let us now see the poset for each of our earlier partial metric spaces. For sequences, $x \sqsubseteq_{d_S} y$ if and only if there exists some $k \le \infty$ such that the length of $x$ is $k$, and for each $i < k$, $x_i = y_i$. In other words, $x \sqsubseteq_{d_S} y$ if and only if $x$ is an initial part of $y$. For example, suppose we wrote a computer program to print out all the prime numbers. Then the printing out of each prime number is described by the chain

$\langle\,\rangle \sqsubseteq_{d_S} \langle 2\rangle \sqsubseteq_{d_S} \langle 2,3\rangle \sqsubseteq_{d_S} \langle 2,3,5\rangle \sqsubseteq_{d_S} \ldots$, whose least upper bound is the infinite sequence $\langle 2,3,5,\ldots\rangle$ of all prime numbers.

For the partial metric max over the nonnegative reals, $\sqsubseteq_{\max}$ is the usual $\geq$ ordering. In $F(X)$ $N_r(x) \sqsubseteq_p N_s(y)$ if and only if $N_s(y) \subseteq N_r(x)$. (Smaller tolerances and smaller sets "know" more.)

Thus the notion of a partial metric extends that of a metric by introducing nonzero self-distance, which can then be used to define the relation "is part of", which, for example, can be applied to model the output from a computer program.

*3: The Contraction Fixed Point Theorem.*

We now consider how a familiar theorem from the theory of metric spaces can be carried over to partial metric spaces. *Complete spaces, Cauchy sequences,* and the *contraction fixed point theorem* are all well known in the theory of metric spaces, and can be generalized to partial metric spaces. The next definition generalizes the metric space notion of Cauchy sequence to partial metric spaces.

**3 Definition.** A sequence $x = (x_n)$ in a partial metric space $(X, p)$ is *Cauchy* if there exists $a \geq 0$ such that for each $r > 0$ there exists $k$ such that for all $n, m > k$, $|p(x_n, x_m) - a| < r$.

In other words, $x$ is Cauchy if the numbers $p(x_n, x_m)$ converge to some $a$ as $n$ and $m$ approach infinity, that is, if $\lim_{n,m\to\infty} p(x_n, x_m) = a$. Note that then $\lim_{n\to\infty} p(x_n, x_n) = a$, and so if $(X, p)$ is a metric space then $a = 0$.

**4 Definition.** A sequence $x = (x_n)$ of points in a partial metric space $(X, p)$ *converges* to $y$ in $X$ if $\lim_{n\to\infty} p(x_n, y) = \lim_{n\to\infty} p(x_n, x_n) = p(y, y)$.

So if a sequence approaches a point then its self-distances approach the self-distance of that point.

**5 Definition.** A partial metric space $(X, p)$ is *complete* if every Cauchy sequence converges.

**6 Definition.** For each partial metric space $(X, p)$, $f : X \to X$ is a *contraction* if there is a $k \in [0, 1)$ so that for all $x, y$ in $X$, $p(f(x), f(y)) \leq kp(x, y)$.

**7 Theorem. [Ma95]** *For a contraction $f$ on a complete partial metric space $(X, p)$ there is a unique $x$ in $X$ so that $f(x) = x$. For it, $p(x, x) = 0$.*

**Proof:** For the last assertion, if $f(x) = x$ then $0 \leq p(x, x) = p(f(x), f(x)) \leq kp(x, x)$, so $p(x, x) = 0$. □

The rest of the proof is done later. The fact that the fixed point has self-distance 0, which is trivial in metric spaces, says here that for a computable function which is a contraction the unique fixed point, which is the program's output, will be totally computed (see [Ma95], [Wa]).

*4: Equivalents For Partial Metric Spaces.*

To discover more about the properties of partial metric spaces we now look at some equivalent formulations:

**8 Definition.** A *weighted metric space* is a metric space $(X, d)$ with a function $|\cdot| : X \to [0, \infty)$, satisfying: $|x| - |y| \leq d(x, y)$ for all $x$ and $y$ in $X$.

Let $(X, d, |\cdot|)$ be a weighted metric space, and let $p(x, y) = \frac{d(x,y)+|x|+|y|}{2}$. Then $(X, p)$ is a partial metric space, and $p(x, x) = |x|$. Conversely, if $(X, p)$ is a partial metric space, then $(X, d_p, |\cdot|)$, where (as before) $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and $|x| = p(x,x)$, is a weighted metric space. So from either space we can move to the other and back again. In a weighted metric space the ordering can be defined by $x \sqsubseteq_p y$ if $|x| = d(x,y) + |y|$.

My favorite formulation comes next:

**9 Definition.** A *quasimetric space* is a pair $(X, q)$ where $q : X \times X \to \mathbb{R}$ satisfies:

Q0: $0 \leq q(x,y)$ (nonnegativity),

Q1: if $x = y$ then $q(x,y) = 0$ (equality implies indistancy), and

Q4: $q(x,z) \leq q(x,y) + q(y,z)$ (triangularity).

A quasimetric space is called $t0$ if it also satisfies:

Q2: if $q(x,y) = q(y,x) = 0$ then $x = y$ (indistancy implies equality).

Unsurprisingly, for a quasimetric space $(X, q)$, $f : X \to X$ is a *contraction* if for some $k \in [0,1)$, $q(f(x), f(y)) \leq kq(x,y)$ for each $x, y \in X$.

As quasimetrics are usually not symmetric, we revise our definition of *indistancy* to be $q(x,y) = q(y,x) = 0$. Thus in $t0$ quasimetric spaces equality is identified with indistancy. Each quasimetric space has a *dual*, $(X, q^*)$, where we define $q^*(x,y) = q(y,x)$, and a *symmetrization*, $(X, q^s)$, where $q^s = q + q^*$. Certainly the dual is a quasimetric space, and the symmetrization is a pseudometric space, and is a metric space if $(X, q)$ is $t0$.

**10 Definition.** A *weighted quasimetric space* is a quasimetric space $(X, q)$ together with a *weight function* $|\cdot| : X \to [0, \infty)$ so that

$$\text{for all } x, y \in X,\ |x| + q(x,y) = |y| + q(y,x).$$

For a weighted quasimetric space, let $p(x,y) = |x| + q(x,y)$; then $(X, p)$ is a partial metric space. For $(X, p)$ a partial metric space, $(X, q_p, |\cdot|_p)$, is a weighted quasimetric space, where $q_p(x,y) = p(x,y) - p(x,x)$ and $|x|_p = p(x,x)$. (So for any partial metric, $q_p^*(x,y) = q_p(y,x) = p(x,y) - p(y,y)$.)

Not every quasimetric space has a weight, $|\cdot|$ (see [Ma94]). But completeness and the contraction fixed point theorem extend easily to quasimetrics:

**11 Definition.** A sequence in a quasimetric space $(X, q)$ is *Cauchy* if for each $r > 0$ there is an $n \in \mathbb{N}$ such that if $n \leq m, p \in \mathbb{N}$ then $q(x_m, x_p) < r$. $(X, q)$ is $^s-$*complete* if each Cauchy sequence has a limit in $(X, q^s)$.

The "switch" from $q$ to $q^s$ is less odd than it might seem. A sequence is Cauchy in $(X, q)$ if and only if it is Cauchy in $(X, q^*)$, if and only if it is Cauchy in $(X, q^s)$: For if it is Cauchy in $(X, q)$, for $r > 0$ find $n \in \mathbb{N}$ so that $q(x_m, x_p) < r$ if $n \leq m, p$; then for the same $n$ and such $m, p$, $q^*(x_m, x_p) = q(x_p, x_m) < r$ and $q^s(x_m, x_p) < 2r$. Conversely if it is Cauchy in $(X, q^*)$ then it is Cauchy in $(X, (q^*)^*) = (X, q)$, and if it is Cauchy in $(X, q^s)$, then for $r > 0$ find $n \in \mathbb{N}$ so that $q^s(x_m, x_p) < r$ if $n \leq m, p$; then for the same $n$ and such $m, p$, $q(x_m, x_p) \leq q^s(x_m, x_p) < r$.

But limits are not the same: $q_{\max}(x,y) = \max\{0, y - x\}$, so $q^s_{\max}(x,y) = \max\{0, y - x\} + \max\{0, x - y\} = |x - y|$, the usual Euclidean metric. In $(\mathbb{R}^+, q_{\max})$, 0 is a $q^*_{\max}$-limit of each sequence, since for each $r > 0$, $q^*_{\max}(0, x_m) = q_{\max}(x_m, 0) = 0 < r$ for each $m \in \mathbb{N}$. Our definitions are the same as for the metric space $(X, q^s)$, no new proof is needed for:

12 Theorem. *For each contraction $f$ on an $^s$-complete $t0$ quasimetric space, there is a unique $x$ in $X$ so that $f(x) = x$.*

## *5: Generalized Metric Topology.*

But I hear a rumbling in the back "aren't limits topological?"

The open balls in a metric space yield a topology called the *metric* topology. This is easily generalized to quasimetric and partial metric spaces:

13 Definition. Given a quasimetric space $(X, q)$, $x \in X$, and $r > 0$, $B_r^q(x) = \{y : q(x, y) < r\}$ is the *open ball* with center $x$ and radius $r$.

For a partial metric $p$, we let $B_r^p(x) = B_r^{q_p}(x)$. Thus for a partial metric $p$, $B_r^p(x) = \{y : p(x, y) < p(x, x) + r\}$.

The usual proof that the open balls in a metric space form a basis for a topology carries over, essentially unchanged, to any quasimetric space. This topology is denoted $\tau_q$ (and $\tau_{q_p}$ is abbreviated to $\tau_p$). In particular, when $(X, d)$ is a metric space then this is the usual open ball topology.

But there is an key difference due to lack of symmetry: for a metric, $\tau_d = \tau_{d^*} = \tau_{d^s}$; for a quasimetric $q$, this does not hold. For example, for $r > 0$, $B_r^{q_{\max}}(x) = \{y : \max\{y - x, 0\} < r\} = \{y : y - x < r\} = \{y : y < x + r\}$, so $\tau_{q_{\max}} = \{(-\infty, a) : a \in \mathbb{R}\}$ and similarly $\tau_{q^*_{\max}} = \{(a, \infty) : a \in \mathbb{R}\}$; $q^s_{\max}$ is the usual metric on $\mathbb{R}$, so $\tau_{q^s_{\max}}$ is its usual topology.

For all quasimetrics $\tau_{q^s}$ is the join, $\tau_q \vee \tau_{q^*}$. Of course $q = q^*$ if and only if $q$ is a pseudometric, and then the topologies $\tau_q$, $\tau_{q^*}$, and $\tau_{q^s}$ are identical. To discuss this array of topologies in general, we need:

**14 Definition.** A *bitopological space* is a triple $(X, \tau, \tau^*)$ such that $\tau$ and $\tau^*$ are topologies.

Bitopological spaces were first introduced in [Ke], and are discussed in our notation in [Ko95]. They occur when there is a lack of symmetry.

Each bitopological space $(X, \tau, \tau^*)$ gives rise to a third topology important in the study of these spaces. It is $\tau^s = \tau \vee \tau^*$, the join of $\tau$ and $\tau^*$; $\tau^s$ is called the *symmetrization* topology.

$B^q_{r/2}(x) \cap B^{q^*}_{r/2}(x) \subseteq B^{q^s}_r(x) \subseteq B^q_r(x) \cap B^{q^*}_r(x)$, so $\tau_{q^s} = \tau_q \vee \tau_{q^*}$ is a pseudometric topology, which is metric in the $t0$ case.

Our definition of partial metric convergence of a sequence $(x_n)$ to a point $y$ is that $\lim_{n\to\infty} p(x_n, y) = \lim_{n\to\infty} p(x_n, x_n) = p(y, y)$. This is equivalent to saying that $\lim_{n\to\infty} q_p(y, x_n) = 0 = \lim_{n\to\infty} q^*_p(y, x_n)$, which happens if and only if for each $r > 0$, eventually $q_p(y, x_n) < r$ and eventually $q^*_p(y, x_n) < r$, that is, if and only if $x_n \to y$ with in both $\tau_{q_p}$ and $\tau_{q^*_p}$, that is, if and only if $x_n \to y$ with respect to $\tau_{q^s_p}$. This further reflects the fact that the quasimetric, partial metric, and metric, contraction fixed point theorems are equivalent.

In the motivating case of $S^\omega$, $x_n \to x$, for $x = (s_1, s_2, ...), x_n \in S^\omega$, if and only if for each positive integer $k$, the initial segment $(s_1, ..., s_k)$ is an initial segment of $x_n$ for large enough $n$. In the other key example of nonempty closed bounded intervals in $\mathbb{R}$, for any real number $\{a\} = [a, a]$ which $= \lim_{n\to\infty}[b_n, c_n]$ if and only if for each $k$, $[b_n, c_n] \subseteq [a - 1/k, a + 1/k]$ for large enough $n$.

The relation $x$ *is part of* $y$ is also topological; in fact, $q(x, y) = 0$ if and only if in $\tau_q$, $x \in \mathrm{cl}(\{y\})$. (Since $q(x, y) = 0 \Leftrightarrow (\forall r > 0)(y \in B^q_r(x)) \Leftrightarrow (\forall T \in \tau_q)(x \in T \Rightarrow y \in T)$.) So $\sqsubseteq_q = \sqsubseteq_{\tau_q}$, $\sqsubseteq_\tau$ the *specialization* of $\tau$, a reflexive, transitive relation.

If $(X,q)$ is a $t0$ quasimetric space, by Q2, $(X,\tau_q)$ is a $T_0$ *space*: $(x \in \mathsf{cl}\{y\} \& y \in \mathsf{cl}\{x\}) \Rightarrow x = y$, so $\sqsubseteq_{\tau_q}$ is a partial order. Of course, for metric (like all $T_1$) topologies, $\sqsubseteq_\tau$ is equality.

Thus unlike metrics, quasimetrics and partial metrics give rise to topologies with general separation properties. But like metric topologies, these topologies are first countable.

Thus we generalize further by considering quasimetrics and partial metrics going into powers of $[0,\infty]$, rather than simply $[0,\infty]$.

We will get our topology by saying that a set is open when it contains a ball of positive radius about each of its points. Three properties of the set $P = \{r : r > 0\}$ of positive reals are centrally important in the use of metrics: for each $a, b, r, s$,

(Pa) if $r \in P$ and $r \leq s$ then $s \in P$,

(Pb) if $r, s \in P$ then for some $t \in P$, $t + t \leq r$ and $t + t \leq s$,

(Pc) if $a \leq b + r$ for each $r \in P$, then $a \leq b$.

If $I$ has at least two elements then $\{r \in [0,\infty]^I : r > 0\}$ fails to satisfy (Pb), but a useful set of positives in $[0,\infty]^I$ is $\sum_I (0,\infty] = \{r \in (0,\infty]^I : \{i : r(i) \neq \infty \text{ is finite}\}\}$. Now we can define:

**15 Definition.** A *value space* is a power $V = (\mathbb{R}^+)^I$, with $\mathbb{R}^+ = ([0,\infty], +, \leq, 0, \infty)$ the extended non-negative reals. (On $(\mathbb{R}^+)^I$ $+$, $\leq$, $0$, $\infty$ are defined coordinatewise). A *set of positives on* $V$ is a nonempty $P \subseteq V$ satisfying (Pa)–(Pc). For a set $X$, $q : X \times X \to V$ is a $V$-metric if it satisfies conditions M0 – M4 of Definiton 1, with $V$-quasimetrics, $V$-partial metrics, etc., similarly defined. A generalized metric space is a $V$-metric space for some value space $V$, together with a set $P$ of positives on $V$.

The usual proof shows that $\{T \subseteq X : \text{for each } x \in T,\ N_r(x) \subseteq T \text{ for some } r \in P\}$ is a topology which we call $\tau_{q,P}$ (or simply $\tau_q$); see [Ko88], [Ko04].

Though $N_r(x) = \{y : q(x,y) \leq r\}$ as expected, we must be careful in defining open balls: for $r \in P$, $x \in X$, $B_r(x) = \{y : \text{for some } s \in P,\ q(x,y) + s \leq r\}$. This reduces to the usual definition in $V = \mathbb{R}^+$.

If $r \in P$, then $s + s \leq r$ for some $s \in P$; for this $s$, $B_s(x) \subseteq N_s(x) \subseteq B_r(x)$, so $T \in \tau_{q,P} \Leftrightarrow$ for each $x \in T$, $B_r(x) \subseteq T$ for some $r \in P$. Also, open balls are open in $\tau_{q,P}$, since if $y \in B_r(x)$ then for some $s, t \in P$, $q(x,y) + s \leq r$ and $t + t \leq s$. By the triangle inequality, if $q(y,z) \leq t$ then $q(x,z) + t \leq q(x,y) + q(y,z) + t \leq q(x,y) + s \leq r$, so $N_t(y) \subseteq B_r(x)$. As a result, $\{B_r(x) : r \in P, x \in X\}$ is a base for $\tau_{q,P}$.

16 Theorem. *(a) A topological space $(X,\tau)$ is completely regular if and only if it arises from a generalized pseudometric space.*

*(b) Every topology arises from a generalized quasimetric space.*

*(c) A topology $\tau_{q,P}$ is $T_0$ if and only if $q$ is $t0$.*

*(d) A topology is $T_0$ if and only if it arises from a $t0$ generalized quasimetric space, if and only if it arises from a generalized partial metric space.*

**Proof:** (a) Let $I$ be a set of continuous functions from $(X,\tau)$ to $\mathbb{R}$ such that whenever $x \in T \in \tau$, there is an $f_{x,T} \in I$ so that $f(x) = 1$ and $f[X \setminus T] = \{0\}$. Let $V = (\mathbb{R}^+)^I$ and $P = \sum_I (0,\infty]$.

Define $d : X \times X \to V$ by $d(x,y)(f) = |f(x) - f(y)|$; we finish by showing $\tau = \tau_{d,P}$: If $x \in T \in \tau$, then $r = r_{f_{x,T}} \in \sum_I (0,\infty]$, where $r_{f_{x,T}}(f) = .5$ and $r_{f_{x,T}}(g) = \infty$ if $g \neq f$; also

$N_r(x) = \{y : d(x,y) \le r\} = \{y : |f(x) - f(y)| \le .5\} \subseteq T$, showing $T \in \tau_{d,P}$. Conversely if $x \in T \in \tau_{d,P}$ then for some $r \in \sum_I (0,\infty]$, $N_r(x) \subseteq T$; for some finite $F \subseteq I$, $r(g) = \infty$ for $g \notin F$, so $N_r(x) = \bigcap_{g \in F} \{y : |g(x) - g(y)| \le r\}$, a $\tau$-neighborhood of $x$, so $T \in \tau$. Thus $\tau$ and $\tau_{d,P}$ have the same elements, so $\tau = \tau_{d,P}$ as required.

(b) Note that for $x \in T \in \tau$, the characteristic function, $\chi_T$ is continuous to $[0,\infty]$ with $\tau_{\max^*} = \{(a,\infty] : a \in [0,\infty]\}$. Also, $\chi_T(x) = 1$, and $\chi_T[X \setminus T] = \{0\}$. Now redo the proof of (a) with $d(x,y)(f) = \max\{f(x) - f(y), 0\}$.

Not much tinkering is needed to show (c) and (d) or corresponding facts for quasiproximities. □

A somewhat different proof shows that all quasiuniformities arise from generalized quasimetrics, but only special ones (called "balanced") arise from generalized partial metrics. Here are examples to show variety in the kinds of concepts that can be modeled by such partial metrics.

Partial metrics were designed to discuss computer programs, and our first example comes from this area. A type of poset $(X, \sqsubseteq)$ called a *domain* has been defined to model computation. We now define it; much more can be learned in [AJ]:

A *directed complete partially ordered set (dcpo)* is a poset, $(Q, \le)$, in which each directed subset $S$ has a supremum $\bigvee S$ (recall that a set $S$ is directed by an order $\le$ if for each $r, s \in S$ there is a $t \in S$ such that $r \le t$ and $s \le t$). For any poset $(Q, \le)$, the *way-below relation* $\ll$ is defined by $b \ll a$ if whenever $a \le \bigvee D$, $D$ directed, then $b \le d$ for some $d \in D$. A dcpo is *continuous* if for each $a \in Q$, $\{b : b \ll a\}$ is a directed set and $a = \bigvee\{b : b \ll a\}$.

The above axioms are best understood by considering the elements of $Q$ as sets of accumulated knowledge, and interpreting $a \le b$ to mean that the knowledge in $b$ implies that in $a$. Then $(Q, \le)$ is a dcpo if, for each directed set of sets of knowledge there is a set containing exactly this knowledge. The example $S^*$ of sequences is a continuous poset; in it, the union of a directed set of sequences is the sequence with them as initial subsequences. The closed balls form a continuous poset: the knowledge that a point is in each of a directed collection of balls is given by the fact that it is in their intersection, which is a closed ball. Like all continuous posets, these are spaces in which information is gathered.

For sequences, $b \ll a$ if and only if $b$ is a finite (initial) subsequence of $a$, and $a$ is clearly the supremum of $\{b : b \ll a\}$, so this example (which abstracts the Turing machine) is a continuous dcpo. For the closed balls, $N_r(x) \ll N_s(y)$ if and only if $N_s(y) \subseteq B_r(x)$, so $\{N_r(x) : N_r(x) \ll N_s(y)\}$ is directed by $\supseteq$ and $N_s(y)$ is its $\supseteq$-supremum, so this is also a continuous dcpo.

Given a poset, its *Scott topology,* $\sigma$, is the one whose closed sets are the lower sets which contain the suprema of their directed subsets. That is, a set $C$ is Scott closed if whenever $x \le y \in C$ then $x \in C$, and whenever $D \subseteq C$ is directed then $\bigvee D \in C$ (assuming $\bigvee D$ exists, as it must for a dcpo).

The Scott topology is seen to be appropriate by thinking of $\le$ as the "knowledge order", with $x \ge y$ meaning that $x$ implies $y$. It is natural to consider a set closed if it contains all objects implied by each of its elements, and whenever it contains increasing amounts of knowledge, it has an element with all this knowledge. For each $x \in Q$, the smallest closed set containing $x$ is $\{y : y \leq x\}$; thus the poset order is the specialization order of the Scott topology, so it can only arise from a metric if $\leq$ is equality.

Due to the lack of symmetry embodied in $\leq$, it helps to consider a second topology: the *lower topology,* $\omega$, whose closed sets are generated by the sets of the form $\{y : y \geq x\}$ for $x \in Q$.

In [Ko04] it is shown that for each continuous dcpo, there is a partial metric into a power of the unit interval, $[0,1]^I$, such that $\tau_p$ is the Scott topology and $\tau_{p^*}$ is the lower topology. Thus the poset order is the specialization order, so in particular $(Q, \leq) = (Q, \leq_p)$.

But many other bitopological spaces can be so represented (to be precise, the ones that so arise are the pairwise Tychonoff spaces; see [Ko04]). It is unclear whether a reasonable characterization of continuous dcpo's can be found in terms of partial metrics.

More traditional examples are found by looking at topologies on $\mathbb{R}^X$, the real valued functions on a set $X$. The best known of these is the topology of uniform convergence, given by the metric $d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$. The partial metric $p_\infty(f,g) = \sup\{\max(f(x), g(x)) : x \in X\}$ gives rise to this topology, since $d_{p_\infty}(f,g) = \|f - g\|_\infty$, the sup norm distance between $f$ and $g$. By earlier discussion, this splits the topology of uniform convergence into two subtopologies: $\tau_{p_\infty}$, its lower open sets, and $\tau_{(p_\infty)^*}$, its upper open sets.

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